

ON AN INTEGRAL OF MARCINKIEWICZ

BY

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Let $f(\theta)$ be a function of period 2π and class L_p , $p > 1$. By $F(\theta)$ is meant a primitive of $f(\theta)$, i.e.,

$$F(\theta) = \int_0^\theta f(x)dx + C.$$

Marcinkiewicz [3] defined the function

$$\begin{aligned}\mu_r(\theta) &= \left\{ \int_0^{2\pi} ([F(\theta + t) + F(\theta - t) - 2F(\theta)]^{r/(r+1)} dt \right\}^{1/r} \\ &= \left\{ \int_0^\pi t^{r-1} \left[\frac{F(\theta + t) + F(\theta - t) - 2F(\theta)}{t^2} \right]^r dt \right\}^{1/r}\end{aligned}$$

and demonstrated the inequalities⁽¹⁾

$$\begin{aligned}\|\mu_p\|_p &\leq A_p \|f\|_p, & p \geq 2 \\ \|f\|_p &\leq A_p \|\mu_p\|_p, & 1 < p \leq 2.\end{aligned}$$

Since addition of a constant to $f(\theta)$ does not alter μ_p , the second inequality clearly requires

$$\int_0^{2\pi} f(\theta) d\theta = 0.$$

Marcinkiewicz raised the question whether $\mu = \mu_2$, i.e.,

$$\mu = \left\{ \int_0^{2\pi} [F(\theta + t) + F(\theta - t) - 2F(\theta)]^2 / t^3 dt \right\}^{1/2},$$

satisfied the inequality

$$A_p \|f\|_p \leq \|\mu\|_p \leq A_p \|f\|_p$$

(the left side requiring, of course, that $\int_0^{2\pi} f d\theta = 0$). This was affirmatively answered by Zygmund in [7], to which paper the reader is referred for an account of the origin of this problem and its relation to other problems in the theory of Fourier series and in the theory of functions.

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⁽¹⁾ Here, as in the sequel, $\|s(t)\|_p$ denotes $\{\int_E |s(t)|^p dt\}^{1/p}$, the range E being suitably defined to meet the exigencies of the particular situation. A constant is denoted by an upper-case latin capital, dependence on a parameter being indicated by a subscript; one letter will be used to denote more than one constant.

The purpose of this paper is to extend this result of Zygmund to the analogously defined function

$$(1) \quad \mu(\tau) = \left\{ \int_0^\infty [F(\tau+t) + F(\tau-t) - 2F(\tau)]^2/t^3 dt \right\}^{1/2},$$

where $F(\tau)$ is a primitive of a function $f(\tau)$ of class L_p , $p > 1$ in $(-\infty, \infty)$. Our result is stated explicitly in the following theorem.

THEOREM. *Let $f(\tau)$ be a function of class L_p , $p > 1$, in $(-\infty, \infty)$ and $\mu(\tau)$ defined as in (1), then*

$$A_p \|f\|_p \leq \|\mu\|_p \leq A_p \|f\|_p.$$

It will be necessary to utilize certain concepts and results of the theory of functions, which will be briefly noted here.

If $f(\tau)$ is of class L_p , $p > 1$, in $(-\infty, \infty)$, its conjugate function $\tilde{f}(\tau)$ is defined by

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{\tau - x} dx;$$

the integration, naturally, is meant in the principal value sense. The function $\tilde{f}(\tau)$ is of class L_p , and [4] we have

$$A_p \|\tilde{f}\|_p \leq \|f\|_p \leq A_p \|\tilde{f}\|_p.$$

$P(\sigma, \tau)$ will denote the Poisson kernel for the half-plane, $\sigma/(\sigma^2 + \tau^2)$, and the Poisson integral

$$\phi(s) = \phi(\sigma + i\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} (f(x) + i\tilde{f}(x)) P(\sigma, \tau - x) dx$$

is an analytic function in $\Re(s) > 0$ and is in class $\mathcal{H}_p$, i.e., the integral

$$\int_{-\infty}^{\infty} |\phi(\sigma + i\tau)|^p d\tau$$

is uniformly bounded in $0 < \sigma < \infty$. If

$$\|\phi(\sigma + i\tau)\|_p = \left\{ \int_{-\infty}^{\infty} |\phi(\sigma + i\tau)|^p d\tau \right\}^{1/p},$$

then $\|\phi\|_p$ will denote $\|\phi(0 + i\tau)\|_p$. It may be shown that

$$\|\phi(\sigma + i\tau)\|_p \rightarrow 0$$

as $\sigma \rightarrow \infty$, continuously and monotonically, that

$$\begin{aligned} |\phi(\sigma + i\tau)| &< C\sigma^{-1/p} \|\phi\|_p, \\ |\phi'(\sigma + i\tau)| &< C\sigma^{-(1+p)/p} \|\phi\|_p, \end{aligned}$$

and also that for $\delta > 0$,

$$\lim_{\rho \rightarrow \infty} |\phi(\delta + \rho e^{i\theta})| = 0, \quad -\pi/2 \leq \theta \leq \pi/2,$$

uniformly in θ [2].

We shall list several lemmas to which we shall refer later.

Let

$$\Phi(\tau) = \sup_{s \in R_\tau} |\phi(s)|$$

where

$$R_\tau = \{s \mid s = i\tau + \rho e^{i\theta}, \rho > 0, -\alpha < \theta < \alpha < \pi/2\}.$$

LEMMA 1. *If ϕ is in class $\mathcal{H}_p$, $p > 1$, then*

$$\|\Phi\|_p \leq A_p \|\phi\|_p.$$

This result is due to Hardy and Littlewood [1] in the case of the circle; the extension to the half-plane is due to E. Trombley [5].

LEMMA 2. *If $\phi(s)$ is in class $\mathcal{H}_p$, $p \geq 1$, then*

$$\phi'(s) = o(1)$$

as $|s| \rightarrow \infty$ uniformly in every half-plane $\sigma > \alpha > 0$.

A proof of this is to be found in a paper of this author [6].

LEMMA 3. *If*

$$g(\tau) = g(\tau; \phi) = \left\{ \int_0^\infty \sigma |\phi'(\sigma + i\tau)|^2 d\sigma \right\}^{1/2}$$

and $\phi(s)$ is in class $\mathcal{H}_p$, $p \geq 1$, then

$$\|g\|_p \leq A_p \|\phi\|_p.$$

LEMMA 4. *If*

$$g^*(\tau) = g^*(\tau; \phi) = \left\{ \frac{1}{\pi} \int_0^\infty \sigma d\sigma \int_{-\infty}^\infty |\phi(\sigma + i\beta)|^2 P(\sigma, \tau - \beta) d\beta \right\}^{1/2},$$

and $\phi(s)$ is in class $\mathcal{H}_p$, $p > 1$, then

$$\|g^*\|_p \leq A_p \|\phi\|_p.$$

These results (together with their converses, which are not required here) are also to be found in [6].

We demonstrate first that

$$\|f\|_p \leq A_p \|\mu\|_p.$$

Let

$$F(\tau) = \int_c^\tau f(t) dt.$$

Then

$$f(\sigma, \tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) P(\sigma, \tau - t) dt = \frac{1}{\pi} \int_{-\infty}^{\infty} F(t) P_\tau(\sigma, \tau - t) dt$$

and

$$f_\tau(\sigma, \tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(t) P_{\tau\tau}(\sigma, \tau - t) dt.$$

We note that $P_{\tau\tau}(\sigma, \tau)$ is even and that $\int_0^\infty P_{\tau\tau} d\tau = 0$. Thus we may write

$$\begin{aligned} f_\tau(\sigma, \tau) &= \frac{1}{\pi} \int_{-\infty}^{\infty} F(\tau + t) P_{tt}(\sigma, t) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} [F(\tau + t) + F(\tau - t) - 2F(\tau)] P_{tt}(\sigma, t) dt. \end{aligned}$$

Let us fix τ . We write

$$F(\tau + t) + F(\tau - t) - 2F(\tau) = \xi(t).$$

Consider

$$\sigma |f_\tau(\sigma, \tau)|^2 \leq A\sigma \left| \int_0^\infty \xi(t) P_{tt}(\sigma, t) dt \right|^2.$$

A simple computation shows that for $|t| < \sigma$

$$|P_{tt}| < C\sigma^{-3},$$

and for $|t| > \sigma$

$$|P_{tt}| < C\sigma t^{-4}.$$

Thus

$$\sigma \left(\int_0^\sigma \xi P_{tt} dt \right)^2 < C\sigma^{-5} \left(\int_0^\sigma |\xi| dt \right)^2 \leq C\sigma^{-4} \int_0^\sigma |\xi|^2 dt$$

and

$$\sigma \left(\int_{\sigma}^{\infty} \xi P_{it} dt \right)^2 < C \sigma^3 \left(\int_{\sigma}^{\infty} |\xi| t^{-4} dt \right)^2 \leq C \int_{\sigma}^{\infty} |\xi|^2 t^{-4} dt.$$

Integrating with respect to σ ,

$$\int_0^{\infty} \sigma^{-4} d\sigma \int_0^{\sigma} |\xi|^2 dt \leq \int_0^{\infty} |\xi|^2 \left(\int_t^{\infty} \sigma^{-4} d\sigma \right) dt = C \mu^2(\tau)$$

and

$$\int_0^{\infty} \sigma d\sigma \int_{\sigma}^{\infty} |\xi|^2 t^{-4} dt \leq \int_0^{\infty} |\xi|^2 t^{-4} \left(\int_0^t d\sigma \right) dt = \mu^2(\tau).$$

Hence, if $w(\tau)$ denotes

$$\left\{ \int_0^{\infty} \sigma |f_{\tau}(\sigma, \tau)|^2 d\sigma \right\}^{1/2}$$

we have shown that

$$w(\tau) \leq C \mu(\tau).$$

We shall now demonstrate that

$$\|f\|_p \leq A_p \|w\|_p$$

and thus the required inequality will be established.

We assume a right translation of the vertical axis so that $f(\tau)$ assumes its original values on $\mathcal{R}(s) = -\eta$, $\eta > 0$.

We define

$$f^*(\tau) = f_N^*(\tau) = |f|^{p-1} (\text{signum } f) \chi_N(\tau),$$

where

$$\chi_N(\tau) = \begin{cases} 1, & |\tau| < N-1, \\ 0, & |\tau| > N, \\ >0, \leq 1, \text{ and is continuous elsewhere.} \end{cases}$$

The following notation is adopted:

$\phi = u + iv$ is the analytic function associated with f ,

$\psi = u^* + iv^*$ is the analytic function associated with f^* ,

$\gamma = g(\tau; \psi)$.

Consider

$$\int_{\epsilon}^K \sigma (u u^*)_{\sigma\sigma} d\sigma = \left[\sigma (u u^*)_{\sigma} \right]_{\epsilon}^K - (u u^*) \Big|_{\epsilon}^K.$$

Using a procedure identical with that given in [6], where this device is employed in proving our theorem on $g(\tau)$, we may show that

$$\int_0^K \sigma(uu^*)_{\sigma\sigma} d\sigma = u(0, \tau)u^*(0, \tau) + o(1) \frac{K}{K^2 + \tau^2}$$

for large K .

Further we may show that for $\Re > N$

$$\int_{-N}^N ff^* d\tau = \int_{-\infty}^{\infty} ff^* d\tau \leq 2 \int_0^K \sigma \left| \int_{-\Re}^{\Re} (u_{\tau} u_{\tau}^* + u_{\sigma} u_{\sigma}^*) d\tau \right| d\sigma + K\epsilon(\Re) + \epsilon(K)$$

where $\epsilon(y)$ denotes a function $o(1)$ as $y \rightarrow \infty$.

We shall show that

$$\int_{-\Re}^{\Re} (u_{\sigma} u_{\sigma}^* - u_{\tau} u_{\tau}^*) d\tau = o(1)$$

as $\Re \rightarrow \infty$ for $\sigma > 0$, that it is bounded by a function of class L_1 , and hence

$$\int_{-\infty}^{\infty} ff^* d\tau \leq 4 \int_0^K \sigma \left| \int_{-\Re}^{\Re} u_{\tau} u_{\tau}^* d\tau \right| d\sigma + K^2 \epsilon(\Re) + \epsilon(K).$$

We note that $(u_{\sigma} u_{\sigma}^* - u_{\tau} u_{\tau}^*) = \Re(\phi' \phi')$. Consider the rectangle Δ with edges

$$\begin{aligned} \alpha: (\sigma - i\Re, \sigma + i\Re), & \quad \beta: (\sigma + i\Re, \sigma + \nu + i\Re), \\ \gamma: (\sigma + \nu - i\Re, \sigma + \nu + i\Re), & \quad \delta: (\sigma - i\Re, \sigma + \nu + i\Re). \end{aligned}$$

Clearly

$$\oint_{\Delta} (\phi' \psi') ds = 0.$$

Thus

$$\left| \int_{-\Re}^{\Re} (u_{\sigma} u_{\sigma}^* - u_{\tau} u_{\tau}^*) d\tau \right| \leq \int_{\gamma} |\phi' \psi'| d\tau + \int_{\beta, \delta} |-u_{\sigma} u_{\tau}^* + v_{\sigma} u_{\sigma}^*| d\sigma.$$

We have

$$\begin{aligned} |u_{\tau}^*| &\leq C \int_{-\Re}^{\Re} \frac{\sigma(\tau - t)}{(\sigma^2 + (\tau - t)^2)^2} dt \\ &\leq NC\sigma \frac{|\tau| + \Re}{(\sigma^2 + (|\tau| - N)^2)^2} \\ &= o(1) \text{ as } |\tau| \rightarrow \infty, \text{ uniformly in } \sigma, \end{aligned}$$

and similarly

$$\begin{aligned}
|u_\sigma^*| &\leq C \int_{-N}^N \frac{|(\tau - t)^2 - \sigma^2|}{(\sigma^2 + (\tau - t)^2)^2} dt \\
&\leq C \int_{-N}^N \frac{1}{\sigma^2 + (\tau - t)^2} dt \\
&\leq CN \frac{1}{\sigma^2 + (|\tau| - N)^2} \\
&= o(1) \text{ as } |\tau| \rightarrow \infty, \text{ uniformly in } \sigma.
\end{aligned}$$

Thus

$$\int_{\beta} |-u_\sigma u_\tau^* + v_\sigma u_\sigma^*| d\sigma \leq \epsilon(\mathfrak{N}) \int_{\beta} |\phi'| d\sigma = \epsilon(\mathfrak{N})$$

since $\phi' = O(\sigma^{-(1+p)/p})$ as $\sigma \rightarrow \infty$ and is bounded in the half-plane (Lemma 2), and similarly for the integral over segment δ . Also

$$\int_{\gamma} |\phi' \psi'| d\tau \leq C \int_{-\mathfrak{N}}^{\mathfrak{N}} (\Phi/\nu) \cdot (\Psi/\nu) d\tau \leq \frac{C}{\nu^2} \|\Phi\|_p \|\Psi\|_{p'} = \frac{C}{\nu^2}.$$

If $\nu = 1/\mathfrak{N}$ we have at once that

$$\int_{-\mathfrak{N}}^{\mathfrak{N}} (u_\sigma u_\sigma^* - u_\tau u_\tau^*) dt = o(1)$$

as $\mathfrak{N} \rightarrow \infty$ independent of σ .

Since

$$\begin{aligned}
\left| \int_{-\mathfrak{N}}^{\mathfrak{N}} (u_\sigma u_\sigma^* - u_\tau u_\tau^*) d\tau \right| &\leq \int_{-\mathfrak{N}}^{\mathfrak{N}} |\phi' \psi'| d\tau \\
&\leq \|\Phi\|_p \|\Psi\|_{p'} \cdot \frac{1}{\sigma^2} \\
&= C \cdot \frac{1}{\sigma^2}
\end{aligned}$$

for large σ , and by the previous argument is bounded for σ small and $\mathfrak{N}$ sufficiently large, clearly it is dominated by a function of class L_1 , and our estimate of $\int_{-\infty}^{\infty} f h d\tau$ has been validated.

Allowing $\mathfrak{N}$ and then K to approach infinity, we have

$$\int_{-\infty}^{\infty} f h d\tau \leq C \int_0^{\infty} \sigma \int_{-\infty}^{\infty} |u_\tau u_\tau^*| d\tau d\sigma$$

and so

$$\begin{aligned}
\int_{-\infty}^{\infty} f h d\tau &\leq C \int_0^{\infty} \sigma \int_{-\infty}^{\infty} |u, \psi'| d\tau d\sigma \\
&\leq C \int_{-\infty}^{\infty} \left(\int_0^{\infty} \sigma |u_{\tau}|^2 d\sigma \right)^{1/2} \left(\int_0^{\infty} \sigma |\psi'|^2 d\sigma \right)^{1/2} d\tau \\
&= C \int_{-\infty}^{\infty} w(\tau) \gamma(\tau) d\tau \\
&\leq C \|w\|_p \|\gamma\|_{p'} \\
&\leq A_p \|w\|_p \|h\|_{p'}
\end{aligned}$$

by Lemma 3. Allowing N to approach infinity we have

$$\int_{-\infty}^{\infty} |f|^p d\tau \leq A_p \|w\|_p \|f\|_p^{p-1}$$

or

$$\|f\|_p \leq A_p \|w\|_{p'}.$$

We recall, however, that a right translation of the vertical axis has been made. For $\eta > 0$,

$$w(\tau; f(\sigma + \eta + i\tau)) \leq w(\tau; f(\sigma + i\tau))$$

and

$$\|f(\eta + i\tau)\|_p \rightarrow \|f(\tau)\|_p$$

as $\eta \rightarrow 0$. Therefore

$$\|f\|_p \leq A_p \|w\|_p$$

where f is as originally defined and w corresponds to it. Since

$$w \leq C_{\mu}$$

our inequality has been established.

We turn now to the inequality

$$\|\mu\|_p \leq A_p \|f\|_p.$$

First we define (analogously to Zygmund [7])

$$F(\sigma, t) = \int_0^t f(\sigma + i\tau) d\tau,$$

$$F_1(\sigma, t) = F(\sigma, \tau + t) + F(\sigma, \tau - t) - 2F(\sigma, \tau),$$

$$F_1(t) = F_1(0, t),$$

$$f_1(\sigma, t) = \partial F_1(\sigma, t) / \partial t = f(\sigma, \tau + t) - f(\sigma, \tau - t).$$

We are then able to write

$$\mu(\tau) = \left\{ \int_0^\infty ([F_1(t)]^2/t^3) dt \right\}^{1/2}.$$

Let

$$F_1(t) = \{F_1(0, t) - F_1(\sigma_t, t)\} + \{F_1(\sigma_t, t) - F_1(\sigma_t, 0)\} = V + W.$$

Consider the function W . Proceeding in a manner similar to that of Zygmund [7], we have

$$W^2 \leq \left\{ \int_0^t |f_1(\sigma_t, u)| du \right\}^2 \leq t^3 \int_{-t}^t |f_\tau(\sigma_t, \tau + v)|^2 dv.$$

Letting $\phi(\sigma + i\tau)$ again be the analytic function associated with $f(\tau)$, and $\sigma_t = t$,

$$\begin{aligned} \int_0^\infty t^{-3} W^2(t) dt &\leq \int_0^\infty d\sigma \int_{-\sigma}^\sigma |\phi'(\sigma + i(\tau + v))|^2 dv \\ &\leq A g^{*2}(\tau). \end{aligned}$$

By an integration by parts

$$|V| = |F_1(0, t) - F_1(\sigma_t, t)| \leq \int_0^{\sigma_t} \sigma |F_{1\sigma\sigma}| d\sigma + \sigma_t |F_{1\sigma}(\sigma_t, t)| = V_1 + V_2.$$

We write

$$\begin{aligned} V_1 &\leq \int_0^{\sigma_t} \sigma |f_\tau(\sigma, \tau + t)| d\sigma + \int_0^{\sigma_t} \sigma |f_\tau(\sigma, \tau - t)| d\sigma + \int_0^{\sigma_t} \sigma |f_\tau(\sigma, \tau)| d\sigma \\ &= V_{11} + V_{12} + V_{13}. \end{aligned}$$

Letting $\sigma_t = t$, we have

$$\begin{aligned} \int_0^\infty V_{11}^2 t^{-3} dt &\leq \int_0^\infty t^{-2} dt \int_0^t \sigma |f_\tau(\sigma, \tau + t)|^2 d\sigma \\ &\leq \int_0^\infty t^{-2} dt \int_0^t \sigma |\phi'(\sigma + i(\tau + t))|^2 d\sigma \leq A g^{*2}(\tau) \end{aligned}$$

and similarly

$$\int_0^\infty V_{12}^2 t^{-3} dt \leq A g^{*2}(\tau), \quad \int_0^\infty V_{13}^2 t^{-3} dt \leq g^2(\tau).$$

Thus

$$\int_0^\infty V_1 t^{-3} dt \leq A \{g^2(\tau) + g^{*2}(\tau)\}.$$

Examining V_2 now,

$$\begin{aligned} V_2 &= \sigma_t \left| F_{1_\bullet}(\sigma_t, t) \right| \\ &= \sigma_t \left| F_\sigma(\sigma_t, \tau + t) + F_\sigma(\sigma_t, \tau - t) - 2F_\sigma(\sigma_t, \tau) \right|. \end{aligned}$$

Note that

$$F_\sigma(\sigma, u) = \int_0^u f_\sigma(\sigma, x) dx = \int_0^u \tilde{f}_x(\sigma, x) dx = \tilde{f}(\sigma, u) - \tilde{f}(\sigma, 0).$$

Hence

$$V_2 = \sigma_t \left| \tilde{f}(\sigma_t, \tau + t) + \tilde{f}(\sigma_t, \tau - t) - 2\tilde{f}(\sigma_t, \tau) \right| \leq \sigma_t \int_{-t}^t \left| \tilde{f}_\tau(\sigma_t, \tau + v) \right| dv,$$

and

$$V_2^2 \leq A \sigma_t^2 \cdot t \int_{-t}^t \left| \phi'(\sigma_t + i(\tau + v)) \right|^2 dv.$$

Letting $\sigma_t = t$,

$$\int_0^\infty V_2^2 t^{-3} dt \leq A \int_0^\infty d\sigma \int_{-\sigma}^\sigma \left| \phi'(\sigma + i(\tau + v)) \right|^2 dv \leq A g^{*2}(\tau).$$

We have then

$$\mu^2(\tau) \leq A \{g^2(\tau) + g^{*2}(\tau)\},$$

and by Lemmas 3 and 4

$$\|\mu\|_p \leq A_p \|f\|_p$$

for $p > 1$, which completes the proof of our theorem.

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